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Nonclassical features of the electromagnetic field

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Abstract : Fields generated in a large class of nonlinear optical processes, including those with losses, have a Wigner distribution that is Gaussian centered around the mean value of the field. Various classical and local inequalities that are violated by quantum fields, such as those generated in nonlinear processes, are expressed as inequalities relating to the parameters of the underlying Wigner distribution function The possibility of distinguishing a quantum system from a classical one in terms of the distribution function parameters is discussed.

Keywords : Nonclassical fields, Wigner function, nonlinear optical processes

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1. Introduction

In the field of quantum optics, contrary to popular belief, classical wave theories provide adequate description of most phenomena. There are only a few examples, *viz.* antibunching [1], sub (or super)-Poissonian photon statistics [2], squeezing [3], quantum interference [4], and spontaneous emission, where one needs to evoke the principles of quantum mechanics to arrive at the result observed in experiments. There is also the case of quantum mechanical nonlocality which has prompted several optical correlation experiments [5,6] to probe what is known as the Einstein–Podolsky–Rosen (E–P–R) paradox. Each of these quantum phenomena is normally expressed by the violation of a particular inequality associated with the quantum state under consideration. Therefore, a natural question arises : Is it possible to obtain a general description, may be in the form of a generalized inequality, for all the known nonclassical and nonlocal effects, and distinguish between the classical and the quantum fields in general ?

There are two quite distinct ways to prescribe a quantum state of an electromagnetic field :

(i) If the average photon occupation number per mode is less than unity, then the field cannot be treated classically for some purposes. This is the most familiar condition based on the correspondence principle.

(ii) We make a diagonal coherent state $|\{\nu\}\rangle$ representation [7] of the density operator $\hat{\rho}$ for the field, by writing

$$\hat{\rho} = \int P(\{\nu\})\, |\{\nu\}\rangle\langle\{\nu\}|\, d\{\nu\}, \tag{1.1}$$

$P(\{\nu\})$ is some weight functional or phase-space density (the Glauber-Sudarshan P function) and the integral is to be taken over all values of the set of complex amplitudes $\{\nu\}$. If $P(\{\nu\})$ is not a probability density, then the state is nonclassical. In general $P(\{\nu\})$ has to be regarded as a generalized function, which may be negative and highly singular. An optical field behaves as a classical wave field in all respects only when both conditions (i) and (ii) are violated.

We note that for a quantum electromagnetic field, complete information is contained in the density matrix $\hat{\rho}$ and information about the statistical properties of the field can be obtained from the moments of the field operators. The quantum condition puts restrictions on these moments which are the parameters of the phase-space distribution function corresponding to the density matrix $\hat{\rho}$. In the case of a quantum field, where no well-behaved $P(\{\nu\})$ function exists, other quasiprobability distributions are often used, as for example, the Q and Wigner functions. All known nonclassical and nonlocal effects can then be expressed in terms of the parameters of the respective distribution function.

For a nonlinear material subject to an electric field E, the susceptibility χ is field-dependent and can be written as a power expansion in E. The induced electric polarization P of the medium is then

$$P(E) = \varepsilon_0\left[\chi^{(1)}E(w) + \chi^{(2)}E^2(w) + \chi^{(3)}E^3(w) + \cdots\right. \tag{1.2}$$

where ε_0 is the dielectric permittivity of vacuum and $\chi^{(i)}$ is the i-th order susceptibility tensor. The $\chi^{(2)}$ processes are three-wave mixing interactions of the type : $w_c \quad w_a + w_b$. The terms involving the third-order susceptibility $\chi^{(3)}$ correspond to four-wave mixing (FWM) interactions of the type : $w_e + w_d \quad w_a + w_b$, w's are the frequencies.

It is known that the strong correlation between the output modes generated in nonlinear processes, such as multimode parametric amplifiers and four-wave mixing, may lead to violation of classical inequalities [8]. For fields generated in a large class of such nonlinear optical processes, including those with losses, the quantum state of the generated radiation corresponds to a Gaussian Wigner function which is centered around the mean value of the field [9]. In this paper we adapt the Wigner function description of quantum states for radiation fields, and the inequalities violated by the quantum radiation field are expressed in terms of the parameters of the Wigner distribution function, *i.e.*, the field is shown to acquire quantum features only for certain ranges of the Wigner parameters [10]. This may provide a unifying description of the various known nonclassical and nonlocal features of the radiation field.

2. The Wigner function

The Wigner function corresponding to the density matrix $\hat{\rho}$ of a single mode of the radiation field (characterized by boson operators $\hat{a}$ and $\hat{a}^+$) is defined [11] as :

$$W(z,z^*) = \pi^{-4} Tr\left[\hat{\rho} \int d^2p \, \exp\left\{-\left[p\left(z^*-\hat{a}^+\right) - p^*(z-\hat{a})\right]\right\}\right], \tag{2.1}$$

where the integral is over the entire complex p-plane. Note that

$$\int W(z, z^*) \, d^2z = 1. \tag{2.2}$$

The expectation value (corresponding to the classical ensemble average) of an arbitrary operator $\hat{O}$ is given as

$$<\hat{O}> = \int W(z, z^*) \, O(z) \, d^2z \equiv \langle O(z) \rangle, \tag{2.3}$$

where $O(z)$ is obtained from (2.1) by replacing $\hat{\rho}$ by $\pi\hat{O}$. For a two-mode case, the Wigner function may be defined similarly.

Consider a field with Gaussian Wigner function :

$$W\left(z,z^*\right) = \left[\pi\left(t^2 - 4\mu^2\right)^{1/2}\right]^{-1} \exp\left[-\left\{\mu(z-z_o)^2 + \mu^*\left(z^* - z_0^*\right)^2 + t\left|z - z_0^*\right|^2\right\} \Big/ \left(t^2 - 4|\mu|^2\right)\right], \tag{2.4}$$

where

$$\langle\hat{a}\rangle = z_0, \tag{2.5}$$

$$\left\langle\hat{a}^2\right\rangle = -2\mu^* + z_0^2, \tag{2.6}$$

$$\left\langle\hat{a}^{+2}\right\rangle = -2\mu + z_0^{*2}, \tag{2.7}$$

$$\left\langle a^+a\right\rangle = t - \frac{1}{2} + |z_0|^2, \tag{2.8}$$

and

$$t > \mu + \mu^*. \tag{2.9}$$

The positive definiteness of the density matrix can be ensured through the following parameterization of μ and t :

$$\mu = Q/4 \sinh x \exp(-i\theta), \tag{2.10}$$

$$t = Q/2 \cosh x, \tag{2.11}$$

with the restriction that

$$O \geq 1. \tag{2.12}$$

The Gaussian Wigner distribution (2.4) is applicable to fields generated in nonlinear optical processes around steady-state, as has been shown in the examples of parametric down-conversion (three-wave mixing) in a cavity [12], the two-photon squeezed laser (four-wave mixing) [13]. The density matrix equation of these examples can be converted into a differential equation for the Wigner distribution function through the Weyl ordering mappings. This equation has the form of a linearized Fokker-Planck equation, the solution of which is a Gaussian.

Fluctuations in the photon-number $\hat{n} \equiv a^{+} a$ can be written in terms of the Wigner parameters as

$$\left\langle \left(\Delta\hat{n}\right)^2 \right\rangle \equiv \left\langle \hat{n}^2 \right\rangle - \left\langle \hat{n} \right\rangle^2 = t^2 + 2|z_0|^2 t - \frac{1}{4} - 2z_0^2\mu - 2z_0^2\mu + 4\mu\mu^* . \tag{2.13}$$

3. Classical inequalities

3.1. Quantum interference :

We would like to point out a basic difference between the classical and the quantum mechanical concepts of interference which is often over looked. From a classical point of view, the superposition of two waves, whether mutually coherent or not, always leads to interference. For time intervals short compared with the coherence time of the two waves, both waves may be regarded as sinusoidal oscillations, and interference fringes should always be observable *in principle* with two mutually incoherent light beams. But in the case of a quantum description of the optical field, the state of the field has to be specified, and then the expectation value (corresponding to the classical ensemble average) of the observable (light intensity in this case) in the state is calculated. Thus when the two beams are statistically independent, the interference disappears as a result of the averaging over a large number of independent states. Therefore it is a requirement of quantum mechanics that the two beams must have well-defined phases, *i.e.*, they should be coherent, if interference fringes are to be observable in the experiment.

It is possible to have fourth-order interference, *i.e.*, spatial modulation of intensity correlations that are fourth-order in field amplitudes, even with independent sources where ordinary second-order interference may not exist. The fourth-order inteference 'fringes', of course, cannot be seen with bare eyes, but can be detected by correlation measurements. These fourth-order interference effects are predicted both classically and quantum mechanically, but there are significant differences between the predictions of the two theories regarding the relative depth of modulation or 'visibility' of the interference pattern. For classical fields there are theoretical upper bounds for the visibility and the quantum mechanical states can be made to violate these new kind of inequalities [4].

Let us consider two polarized, approximately plane quasi-monochromatic electromagnetic waves emerging from two points A and B described by complex scalar amplitudes $E_A(t)$ and $E_B(t)$, which are superposed at the receiving plane. Let x_1, x_2 be the positions of the two detectors at the interference plane. If E_A and E_B are random and uncorrelated, the ensemble averages $\langle I(x_1, t)\rangle$ and $\langle I(x_2, t)\rangle$ exhibit no second-order interference. If we evaluate the intensity cross-correlation function $\langle I(x_1, t)\, I(x_2, t)\rangle$ under the same assumption that the two light beams are independent and the phases of E_A, E_B are random, then

$$\langle I(x_1,t)\, I(x_2,t)\rangle = \left\langle (I_A + I_B)^2 \right\rangle [1 + \sigma \cos\{2\pi(x_1 - x_2)/L\}], \tag{3.1}$$

with $L = \lambda D/s = \lambda/\theta$, where λ is the wavelength, D is the distance from the source to the interference plane, s is the separation between the sources and θ is the small angle of inclination between the two light paths from A and B. Eq. (3.1) represents a form of interference, involving correlation function of fourth-order, the periodicity ('fringe'-spacing) of the interference pattern being L. The relative modulation amplitude or 'visibility' σ of the interference pattern is given by

$$\sigma = \frac{2\langle I_A\rangle\langle I_B\rangle}{\left\langle (I_A + I_B)^2 \right\rangle} = \frac{2\langle I_A\rangle\langle I_B\rangle}{(\langle I_A\rangle + \langle I_B\rangle)^2 + \left\langle (\Delta I_A)^2\right\rangle + \left\langle (\Delta I_B)^2\right\rangle}. \tag{3.2}$$

Here $\left\langle (\Delta I)^2 \right\rangle \equiv \left\langle I^2 \right\rangle - \langle I\rangle^2$ gives the fluctuation in I. From eq. (3.2) we see that the intensity cross-correlation is smallest when $|x_1 - x_2| = (n + 1/2)L$, $n = 0, 1, 2, \ldots$, but it can never vanish. The visibility σ in the classical case has a maximum possible value of 1/2, when $\langle I_A\rangle = \langle I_B\rangle$ in the absence of any fluctuations of I_A and I_B, *i.e.*,

$$\sigma \le \frac{1}{2}. \tag{3.3}$$

Let us now consider a specific example of a quantum mechanical source for two-photon interference, a photon-pair created by the nonlinear process of spontaneous parametric frequency down-conversion. In this process photons in the pump laser beam spontaneously 'split' into pairs of lower-frequency signal and idler photons that emerge from the nonlinear medium within a cone around the pump beam axis. For an interaction to take place with appreciable probability, the phase-matching conditions (energy and momentum conservation laws) are to be satisfied and this can be achieved in a uniaxial noncentrosymmetric crystal exhibiting birefringence. If the two down-converted signal and idler beams are recombined at some distant point from which the pump is excluded, we may take the resulting two-photon state to be a linear superposition state. In that state the single-photon detection probability $P(r,t)$ does not exhibit interference fringes and this simply reflects the absence of a phase relation between the signal and idler-waves.

The joint probability of detecting one photon at x_1 and another at x_2 in the interference plane is given by the fourth-order correlation function (normally-ordered) and when signal and idler photons are degenerate and similarly polarized, it comes out to be of the form [4b] ;

$$P_2(x_1,t_1;x_2,t_2) \equiv \langle :\hat{I}(x_1,t_1)\hat{I}(x_2,t_2): \rangle \propto [1+\cos\{2\pi(x_1-x_2)/L\}] \ , \quad (3.4)$$

where L is the spacing of the interference fringes as before. There is a cosine modulation of P_2 (or the joint probability of two-photon detection) with the separation $(x_1 - x_2)$, with periodicity L. The joint probability vanishes when $|x_1 - x_2|$ is an odd integral multiple of half fringe-spacing, and the relative modulation amplitude or 'visibility' σ of the fringe pattern obtained from eq. (3.4) is 1C0 %, unlike the classical situation described by eq. (3.2) where $\sigma \leq 50$ %.

The first observation of this nonclassical effect was reported in 1987 [4a] in an interference experiment involving the down-converted photon-pairs. The results supported the quantum mechanical theory, violating the classical inequality (3.3) with 92% confidence level. The Wigner distribution function of the fields produced by parametric down-conversion is a Gaussian and it is clear that violation of the classical inequality (3.3) can be recast in the form of an inequality involving the Wigner parameters (μ, μ^*, t).

3.2. *Sub (super)-Poissonian statistics :*

When completely coherent light falls on a photoelectric detector, the number of photoelectric counts n registered in some finite time interval obeys Poisson statistics for which the variance $\langle(\Delta n)^2\rangle$ of n equals the mean number $\langle n \rangle$. For classical waves, in general, $\langle(\Delta n)^2\rangle > \langle n \rangle$, as a consequence of intensity fluctuations. However, there exist quantum states of the electromagnetic field for which the photon statistics is sub-Poissonian, *i.e.*,

$$\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle < 0. \quad (3.5)$$

These states have no classical description. From (2.12), inequality (3.5) can be written in terms of the Wigner parameters as

$$t^2 + 2|z_0|^2 t + 1/4 - 2z_0^{*2}\mu^* - 2z_0^2\mu + 4|\mu|^2 - |z_0|^2 - t < 0. \quad (3.6)$$

Similarly, for super-Poissonian photon statistics, we get

$$t^2 + 2|z_0|^2 t + 1/4 - 2z_0^{*2}\mu^* - 2z_0^2\mu + 4|\mu|^2 - |z_0|^2 - t \geq 0. \quad (3.7)$$

It also follows immediately that for $z_0 = 0$ (below threshold),

$$\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle = \left(t - \frac{1}{2}\right)^2 + 4|\mu|^2 \geq 0, \quad (3.8)$$

and so the photon statistics cannot be sub-Poissonian.

3.3. Squeezing :

The coherent state [7] of the radiation field is the closest counterpart to a classical electromagnetic field and is defined as that whose uncertainty product $\Delta E \,.\, \Delta H$ for the electric and magnetic fields is minimum for all time when subject to the simple harmonic potential characteristic of the field. The corresponding wave-packet 'coheres' (does not spread) in time. The coherent state $|v>$ is the right eigenstate of the annihilation operator $\hat{a}$:

$$\hat{a}\ |v\rangle \ = \ v\ \ |v\rangle, \tag{3.9}$$

where v is the complex amplitude.

An ideal laser operating in a pure coherent state would possess quantum noise due to photon-number fluctuations (shot-noise). The electric field operator associated with a mode of angular frequency w at a given position is

$$\hat{E}(t) \ = \ E_0 \left[\hat{X}_1 \ \cos(wt) \ + \ \hat{X}_2 \ \sin(wt)\right], \tag{3.10}$$

where E_0 is a constant, and the quadrature field operators $\hat{X}_1 = i\left(\hat{a}^+ - \hat{a}\right)$ and $\hat{X}_2 = \left(\hat{a}^+ + \hat{a}\right)$ are analogous to the position and momentum operators of a simple harmonic oscillator with $\left[\hat{X}_2,\ \hat{X}_1\right] \ = \ 2i$. The fluctuations in $\hat{X}_1$, $\hat{X}_2$ obey the uncertainty relation :

$$\left\langle\left(\Delta\hat{X}_1\right)^2\right\rangle \left\langle\left(\Delta\hat{X}_2\right)^2\right\rangle \ \geq \ 1. \tag{3.11}$$

The equality sign holds for the minimum uncertainty states (*e.g.*, the coherent state), for which

$$\left\langle\left(\Delta\hat{X}_1\right)^2\right\rangle \ = \ \left\langle\left(\Delta\hat{X}_2\right)^2\right\rangle \ = \ 1. \tag{3.12}$$

Also, a laser is not truly monochromatic, but has a natural linewidth of finite order, arising from phase-diffusion noise due to spontaneous emission fluctuations. These quantum noise set a fundamental limit on the precision of optical interferometric measurements that can be achieved with the use of ordinary lasers.

Squeezed states [3] of the electromagnetic field are a unique set of quantum states (which may or may not be minimum uncertainty states) with less fluctuations in one (i-th) quadrature phase than a coherent state at the expense of increased fluctuations in the other quadrature phase, *i.e.*,

$$\left\langle\left(\Delta\hat{X}_i\right)^2 \ \ < \ 1, \qquad\qquad i \ = \ 1 \text{ or } 2. \tag{3.13}$$

A phase-sensitive nonlinear interaction in a medium is required to generate squeezed states.

In the example of the two-photon squeezed laser [13], the ordinary gain medium inside the laser cavity is replaced by a suitable active nonlinear medium. An intense pump *laser beam causes two-photon excitations in the medium and generates two radiation fields due to four-wave mixing (FWM). The generated photons can get reabsorbed by a two-photon absorption (TPA) process.* A strong competition among FWM, TPA and linear cavity losses leads to lasing action above a certain threshold determined by the nonlinear mixing and the linear damping constants. This is an example of a laser where amplification is obtained without population inversion. As mentioned earlier, in the steady-state, the Wigner distribution function for the generated fields in this case is a Gaussian.

The two photons generated inside the cavity in this process are strongly coupled, as they are either produced simultaneously in FWM or absorbed simultaneously in TPA process, and the phase correlations between them leads to a narrower linewidth of the two-photon laser. The spectrum of fluctuations in the intensity difference between the two output modes shows evidence of strong squeezing, as the photon-number fluctuations of the two modes try to balance each other.

Let us now see how the condition for squeezing of fields can be expressed as an inequality involving the Wigner parameters. The component $\hat{X}_2 = \left(\hat{a} + \hat{a}^{+}\right)$ will be squeezed if the inequality (3.13) holds with $i = 2$, *i.e.*, if

$$\left\langle \Delta\left(\hat{a} + \hat{a}^{+}\right)^{2}\right\rangle < 1. \tag{3.14}$$

For a field described by a Gaussian Wigner function, this leads to the following condition [9] on the parameters μ, μ^{*} and t of the distribution (2.4) :

$$0 < t - \mu - \mu^{*} < 1/2. \tag{3.15}$$

3.4. Nonlocality :

As is well known, the wave-function description of quantum mechanics does not provide the detailed space-time behaviour of a system between the initial preparation and the interaction with the measurement apparatus. This aspect of the quantum measurement process was first discussed by Einstein, Podolsky and Rosen (E–P–R) who concluded that quantum mechanics fails to give an adequate description of physical reality and that in quantum mechanics the motion of a particle must be described in terms of probabilities only because some 'hidden parameters' that determine the motion have not yet been specified.

Quantum theory makes certain predictions that are incompatible with any realistic, local theory. Realism assumes existence of an objective reality independent of whether someone observes it or not. Locality assumes that forces or information can only travel between bodies at speeds less than or equal to that of light. Using essentially the same postulates as those of E–P–R, Bell and several other workers formulated some inequalities obeyed by every realistic, local theory and violated by quantum mechanics. These provide a way to test experimentally the predictions of the local deterministic hidden variable theories against the predictions of quantum mechanics. The possibility of violation of Bell inequalities in correlated states produced by nonlinear processes, such as multimode parametric amplifiers and four-wave mixing, have been studied [8]. The nonlocal character of the generated quantum fields is considered by superposing them with the help of a beam-splitter and it is shown that [10] if one performs a polarization correlation experiment, the violation of the Bell inequality can be achieved for a certain range of the parameters of the Gaussian Wigner distribution of the fields. Violation of the Bell inequality in correlation measurements of mixed signal and idler photons produced in the process of parametric down-conversion has been experimentally observed [6]. Similar violation is predicted in the output of the two-photon squeezed laser described above [13].

Let $\hat{a}$ and $\hat{b}$ be two correlated modes with wave-vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ coming out of a nonlinear material. These are made to fall from opposite sides on a beam-splitter. $\hat{d}$ and $\hat{c}$ are the mixed beams which arrive at the detectors placed at points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ with two polarizers set at variable angles θ_1 and θ_2 in front of them, respectively. The Bell inequality in this case has the following well known form :

$$S \cong P(\theta_1, \theta_2) - P\left(\theta_1, \theta_2'\right) + P\left(\theta_1', \theta_2\right) + P\left(\theta_1', \theta_2'\right) - P\left(\theta_1', -\right) - P(-, \theta_2) \leq 0. \tag{3.16}$$

Here, $P(\theta_1, \theta_2)$ is the joint probability density of detecting two photons for polarizer settings of θ_1 and θ_2 measured by the coincidence counter. $P(\theta_1, -)$ stands for the probability when the second polarizer is removed. Now, the joint probability density of detection of two photons is given as :

$$P(\theta_1, \theta_2) = K\left\langle \hat{d}^+ \hat{c}^+ \hat{c}\, \hat{d} \right\rangle, \tag{3.17}$$

where K is a constant characterizing the detectors. One may write the fields at the detectors as

$$\hat{d}(\boldsymbol{r}_1, \theta_1) = X_a \hat{a} + X_b \hat{b}, \tag{3.18a}$$

$$\hat{c}(\boldsymbol{r}_2, \theta_2) = Y_a \hat{a}' + Y_b \hat{b}, \tag{3.18b}$$

where

$$|X_a|^2 + |X_b|^2 = |Y_a|^2 + |Y_b|^2 = 1. \tag{3.18c}$$

For the correlation experiments [6] with the down-converted signal and idler beams, where $\hat{a}$ and $\hat{b}$ are x-and y-polarized respectively,

$$X_a = i \cos\theta_1 \sqrt{R_x} \exp\left(i\boldsymbol{k}_1' \cdot \boldsymbol{r}_1\right),$$

$$
\begin{aligned}
X_b &= \sin\theta_1 \sqrt{T_y}\ \exp(i\boldsymbol{k}_2\cdot\boldsymbol{r}_1)\,,\\
Y_a &= \cos\theta_2 \sqrt{T_x}\ \exp(i\boldsymbol{k}_1\cdot\boldsymbol{r}_2)\,,\\
Y_b &= -i\ \sin\theta_2 \sqrt{R_y}\ \exp(i\boldsymbol{k}_2'\cdot\boldsymbol{r}_2)\,.
\end{aligned}
\tag{3.19}
$$

We assume that the state of the incident field is such that expectations of unpaired operators vanish. For $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ parallel to $\boldsymbol{k}_2'$ and $\boldsymbol{k}_1'$ respectively, from (3.17) we get

$$
\begin{aligned}
P(\theta_1,\theta_2) = K\Big[&R_xT_x\cos^2\theta_1\cos^2\theta_2\ \langle\hat{n}_a(\hat{n}_a-1)\rangle\\
&+R_yT_y\sin^2\theta_1\sin^2\theta_2\ \langle\hat{n}_b(\hat{n}_b-1)\rangle + \Big(\sqrt{T_x}\sqrt{T_y}\sin\theta_1\cos\theta_2\\
&+\sqrt{R_x}\sqrt{R_y}\cos\theta_1\sin\theta_2\Big)^2\ \langle\hat{n}_a\hat{n}_b\rangle\Big]\,,
\end{aligned}
\tag{3.20}
$$

where $\hat{n}_a \equiv \hat{a}^+\hat{a}$ and $\hat{n}_b \equiv \hat{b}^+\hat{b}$ are the photon-number operators for the two beams incident on the beam-splitter. The probability density when the second polarizer is removed is calculated using unitarity :

$$
\begin{aligned}
P(\theta_1,-) &= P(\theta_1,\theta_2) + P(\theta_1,\theta_2+\pi/2)\\
&= K\ \Big[R_xT_x\cos^2\theta_1\ \langle\hat{n}_a(\hat{n}_a-1)\rangle + R_yT_y\sin^2\theta_1\ \langle\hat{n}_b(\hat{n}_b-1)\rangle\\
&+\ \big\{T_xT_y\sin^2\theta_1 + R_xR_y\cos^2\theta_1\big\}\ \langle\hat{n}_a\hat{n}_b\rangle\Big]\,.
\end{aligned}
\tag{3.21}
$$

A similar expression can be obtained for $P(-,\theta_2)$. For comparison of the different probabilities, all of them should be scaled by the joint probability density when both polarizers are removed :

$$
\begin{aligned}
P(-,-) &= P(\theta_1,-) + P(\theta_1+\pi/2,-) = K\ \big[R_xT_x\langle\hat{n}_a(\hat{n}_a-1)\rangle\\
&+ R_yT_y\langle\hat{n}_b(\hat{n}_b-1)\rangle + \big(T_xT_y + R_xR_y\big)\langle\hat{n}_a\hat{n}_b\rangle\big]\,.
\end{aligned}
\tag{3.22}
$$

Let $R_x = T_x = R_y = T_y = 1/2$, and choose angles $\theta_1 = \pi/8$, $\theta_1' = 3\pi/8$, $\theta_2 = \pi/4$, $\theta_2' = 0$. Then from (3.16), we get

$$
4S/K = -0.85[\langle\hat{n}_a(\hat{n}_a-1)\rangle + \langle\hat{n}_b(\hat{n}_b-1)\rangle] + 0.41\langle\hat{n}_a\hat{n}_b\rangle\,. \tag{3.23}
$$

The Bell inequality (3.16) is violated whenever $4S/K > 0$, *i.e.*,

$$
\frac{\langle\hat{n}_a(\hat{n}_a-1)\rangle + \langle\hat{n}_b(\hat{n}_b-1)\rangle}{\langle\hat{n}_a\hat{n}_b\rangle} < 0.48\,. \tag{3.24}
$$

For optimum choice of angles, the right-hand-side of (3.24) can be made equal to 0.5. The inequality (3.24) can be written [10] in terms of the parameters of the joint Wigner distribution function for the modes $\hat{a}$ and $\hat{b}$.

For the parametric down-conversion process, the photon statistics is nearly Poissonian with mean $\langle \hat{n} \rangle$:

$$\langle \hat{n}_a(\hat{n}_a - 1)\rangle = \langle \hat{n}_b(\hat{n}_b - 1)\rangle = \langle \hat{n}\rangle^2, \tag{3.25a}$$

$$\langle \hat{n}_a \hat{n}_b \rangle = \langle \hat{n}\rangle + \langle \hat{n}\rangle^2. \tag{3.25b}$$

Hence, from (4.9) we get

$$\frac{\langle \hat{n}\rangle^2}{\langle \hat{n}\rangle + \langle \hat{n}\rangle^2} < 0.24, \tag{3.26a}$$

or

$$\langle \hat{n}\rangle < 0.32. \tag{3.26b}$$

Hence from (2.8), the condition for violation of the Bell inequality in this case can be expressed as

$$t + |z_0|^2 < 0.82. \tag{3.27}$$

Once this condition holds, the fields are surely nonlocal.

4. Summary

It is our contention that instead of having different inequalities to describe different quantum features of the electromagnetic field, one can identify a generalized description pertaining to the basic definition of a quantum field, namely in terms of the nonclassical distribution function of the field. From the results of Agarwal and Adam [9] regarding the Wigner distribution function generated in a large class of nonlinear processes producing correlated emeissions, we have given [10] a description of the various nonclassical and nonlocal features of the electromagnetic field in terms of the corresponding Wigner parameters.

References

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